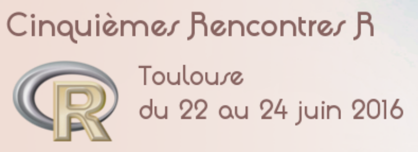
MBBEFD: MODÉLISATION DES TAUX DE DESTRUCTION EN ACTUARIAT NON VIE



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1 INTRODUCTION
- Insurance context

2 MIXED-TYPED DISTRIBUTIONS
- One-inflated distributions
- the MBBEFD distribution

3 PACKAGE MBBEFD
- Characterizing functions
- Fitting methods

4 NUMERICAL ILLUSTRATIONS
- Simulation
- Real dataset

5 CONCLUSION
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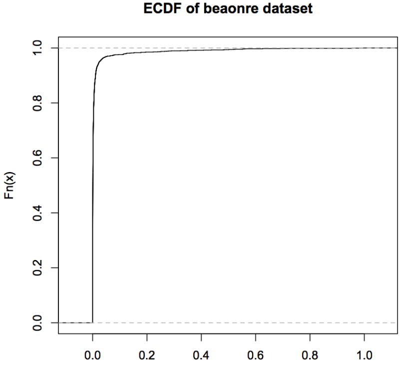
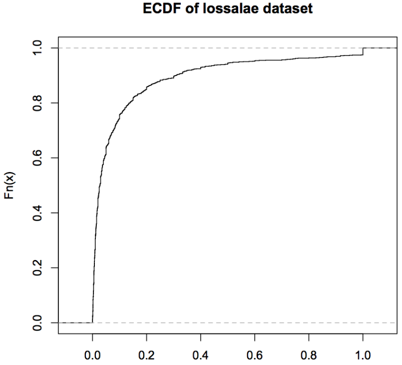
Insurance context

Why model destruction rates?

- To make risks comparable, we look at ratio of losses to the underlying exposure.
- Typically, exposure is the sum insured (SI), the total insured value (TIV), or the maximum probable loss (MPL).
- Shifting the metric from loss amounts to damage ratio allows to benchmarks risks (i.e. insurance guarantees).





Reminder: a unified approach with fitdistrplus

Functionalities of the **fitdistrplus** package

- MLE: Extends the `fitdistr` function with fixed arguments, custom optimization algorithms, possible censoring,
- MME: Provides a generic function to perform moment matching estimation with the raw or centered moments,
- QME: Based on the `quantile` function, provides the quantile matching estimation,
- MGE: Maximum goodness-of-fit is now available with the usual statistical distance and their variants.

So we can fit any probability distribution.

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One-inflated distributions

Let F_0 be a continuous c.d.f. of a variable X_0. The corresponding c.d.f.[1] of the one-inflated random variable X_1 is

$$F_1(x) = (1-p_1)F_0(x) + p_1 1\!\!1_{[1,+\infty[}(x). \quad (1)$$

The density of X_1 should be considered w.r.t. the measure $\mu(x) = \lambda(x) + \delta_1(x)$ for which

$$f_1(x) = p_1^{1\!\!1_{x=1}} (1-p_1)^{1\!\!1_{x\neq 1}} f_0(x)^{1\!\!1_{x\neq 1}}. \quad (2)$$

The quantile function can be derived[2]

$$Q_1(p) = \begin{cases} Q_0(p/(1-p_1)) & \text{if } p < 1-p_1 \\ 1 & \text{if } p \geq 1-p_1 \end{cases} \quad (3)$$

where $Q_0 = F_0^{-1}$ denotes the quantile function of X_0.

[1] no density w.r.t. the Lebesgues measure but an improper density $(1-p)F_0'(x)$ and a probability mass p_1 at $x=1$.

[2] Assuming X_0 is valued on the unit-interval (i.e. $F_0(1)=1$).

Example of one-inflated distributions and estimation

- one-inflated beta:

$$F_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ I(x; a, b)(1 - p_1) & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

where $I(x; a, b)$ denotes the incomplete beta ratio function. The improper density function is $f_1(x) = (1 - p_1)\frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)}$. The expectation is

$$E(X_1) = p_1 + (1 - p_1)\frac{a}{a+b}.$$

Proposition

- *The maximum likelihood estimator of p_1 is the empirical proportion of total loss $\hat{p}_1 = \frac{1}{n}\sum_{i=1}^{n} 1\!\!1_{X_i=1}$, whereas the estimator $\hat{\theta}$ is obtained separately.*
- *$(\hat{p}_1, \hat{\theta})$ converges in distribution to (p_1, θ) as*

$$\sqrt{n}\left(\begin{pmatrix}\hat{p}_1 \\ \hat{\theta}\end{pmatrix} - \begin{pmatrix}p_1 \\ \theta\end{pmatrix}\right) \xrightarrow[n\rightarrow+\infty]{\mathcal{D}} \mathcal{N}_{s+1}(0, \tilde{I}(p_1, \theta)^{-1}),\ \tilde{I}(p_1, \theta) = \begin{pmatrix}\frac{1}{p_1(1-p_1)} & 0 \\ 0 & I_0(\theta)\end{pmatrix}.$$

where $\tilde{I}(p_1, \theta)$ is the Fisher information matrix.

Exposure curves

The exposure curve function of X is defined as the ratio of the limited expected value and the expectation

$$G_X(d) = \frac{E(\min(X,d))}{E(X)}, \; d \in [0,1].$$

- exposure curves are concave increasing.
- direct link between the c.d.f. and the exposure curve :

$$G_X(d) = \frac{\int_0^d (1 - F_X(x))dx}{\int_0^1 (1 - F_X(x))dx} \Leftrightarrow F_X(x) = \left(1 - \frac{G'_X(x)}{G'_X(0)}\right) 1\!\!1_{[0,1[}(x) + 1\!\!1_{[1,+\infty[}(x).$$

Examples

- uniform sur $[0,1]$: $G_X(d) = d(2-d)$.
- Dirac en 1 : $G_X(d) = d$.

MBBEFD(A,B)

Introduced by [B97], the MBBEFD distribution is defined by the following exposure curve for $(a, b) \in \mathcal{D}_{a,b}$

$$\forall x \in I,\ G_X(x) = \begin{cases} \frac{\ln(\frac{a+b^x}{a+1})}{\ln(\frac{a+b}{a+1})} & \text{if } a(1-b) > 0 \\ \frac{1-b^x}{1-b} & \text{if } a = +\infty \text{ and } b < 1 \\ x & \text{if } a = 0 \text{ or } b = 1. \end{cases} \quad (4)$$

where the parameter domain is $\mathcal{D}_{a,b} = \{(a, b), a + 1 > 0, a(1-b) \geq 0, b > 0\}$.

NB: *MBBEFD* stands for Maxwell Boltzmann Bose Einstein Fermi Dirac

D,P,Q FUNCTIONS

The distribution function is

$$\forall x \in I,\ F_X(x) = \begin{cases} \left(1 - \frac{(a+1)b^x}{a+b^x}\right) 1\!\!1_{[0,1[}(x) + 1\!\!1_{[1,+\infty[}(x) & \text{if } a(1-b) > 0 \\ (1-b^x) 1\!\!1_{[0,1[}(x) + 1\!\!1_{[1,+\infty[}(x) & \text{if } a = +\infty \text{ and } b < 1 \\ 1\!\!1_{[1,+\infty[}(x) & \text{if } a = 0 \text{ or } b = 1. \end{cases}$$

A mixed-type distribution with

- a probability mass $P(X=1) = \frac{(a+1)b}{a+b}$,
- an improper density

$$\tilde{f}_X(x) \begin{cases} -\frac{a(a+1)b^x \ln(b)}{(a+b^x)^2} 1\!\!1_{[0,1[}(x) & \text{if } a(1-b) > 0 \\ -\ln(b) b^x 1\!\!1_{[0,1[}(x) & \text{if } a = +\infty \text{ and } b < 1 \\ 0 & \text{if } a = 0 \text{ or } b = 1. \end{cases}$$

PROPOSITION

The MBBEFD(a, b) verifies the regularity and differentiability conditions w.r.t. μ of Theorem 6.5.1 of [CL98], so the MLE converges (in distrib.) to the true value.

MBBEFD(G,B)

Let h be the function $h : (a, b) \mapsto \begin{pmatrix} \frac{a+b}{b(a+1)} \\ b \end{pmatrix}$. h is a bijection from $\mathcal{D}^i_{a,b}$ to $\mathcal{D}^i_{g,b}$ where

$$\mathcal{D}^1_{g,b} = \{(g, b) \in (1, +\infty)^2, bg > 1\}, \ \mathcal{D}^2_{g,b} = \{(g, b) \in (1, +\infty) \times (0, 1), bg < 1\}.$$

Hence, the new parametrization *MBBEFD*(g, b) is defined as

$$G_X(x) = \begin{cases} \frac{\ln\left(\frac{(g-1)b}{1-b} + \frac{1-gb}{1-b} b^x\right)}{\ln(gb)} & \text{if } g > 1, b \neq 1, b \neq 1/g \\ \frac{\ln(1+(g-1)x)}{\ln(g)} & \text{if } g > 1, b = 1 \\ \frac{1-b^x}{1-b} & \text{if } g > 1, bg = 1 \\ x & \text{if } g = 1 \text{ or } b = 0 \end{cases} \quad (5)$$

The distribution function (and other functions) can be derived

$$F_X(x) = \begin{cases} \left(1 - \frac{1-b}{(g-1)b^{1-x}+1-gb}\right) 1\!\!1_{[0,1[}(x) + 1\!\!1_{[1,+\infty[}(x) & \text{if } g > 1, b \neq 1, b \neq 1/g \\ (1 - b^x) 1\!\!1_{[0,1[}(x) + 1\!\!1_{[1,+\infty[}(x) & \text{if } g > 1, bg = 1 \\ 1\!\!1_{[1,+\infty[}(x) & \text{if } g = 1 \text{ or } b = 0 \end{cases}$$

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D, P, Q, R FUNCTIONS FOR ONE-INFLATED DISTRIBUTIONS

- generic one-inflated distributions:

```
doifun(x, dfun, p1, log=FALSE, ...)
poifun(q, pfun, p1, lower.tail = TRUE, log.p = FALSE, ...)
qoifun(p, qfun, p1, lower.tail = TRUE, log.p = FALSE, ...)
roifun(n, rfun, p1, ...)
```

- specific one-inflated distribution : `<d,p,q,r>d` for `d=unif`, `stpareto`, `beta`, `gbeta`.
- exposure curves via `ec` function, e.g. `ecoibeta`
- moments via `m` function, e.g. `moibeta`
- total loss via `tl` function, e.g. `tloibeta`

D, P, Q, R FUNCTIONS FOR THE MBBEFD DISTRIBUTION

- 1st parametrization:

```
dmbbefd(x, a, b, log=FALSE, g)
pmbbefd(q, a, b, lower.tail = TRUE, log.p = FALSE, g)
qmbbefd(p, a, b, lower.tail = TRUE, log.p = FALSE, g)
rmbbefd(n, a, b)
```

- 2nd parametrization: `<d,p,q,r>MBBEFD`
- exposure curves via `ec` function, e.g. `ecmbbefd`
- moments via `m` function, e.g. `mmbbefd`
- total loss via `tl` function, e.g. `tlmbbefd`

(MAXIMUM LIKELIHOOD) ESTIMATION

`fitDR()`

- provides MLE for the following list of distributions: `unif`, `stpareto`, `beta`, `gbeta`, `mbbefd`, `MBBEFD`.
- generates an object of class `"fitDR"` inheriting from the class `"fitdist"`.
- has access to all summarizing functions from **fitdistrplus**: `print`, `summary`, `logLik`, `coef`, `vcov`, `gofstat`,
- has access to all plotting functions from **fitdistrplus**: `cdfcomp, qqcomp, ppcomp, denscomp`.
- has access to bootstrap functions from **fitdistrplus**: `bootdist` and its generic functions,
- provides also total-loss-moment matching estimation.

`eecf()`

- produces an object of class `"eecf"`, `"function"`
- has generic functions `print`, `summary`, `plot`
- `eccomp()` plot exposure curves of multiple fits.

EXAMPLE

```
> x <- roibeta(1e3, 3, 2, 1/6)
> f1 <- fitDR(x, "oibeta", method="mle")
  shape1   shape2
3.032457 2.029166
  shape1   shape2       p1
3.032457 2.029166 0.167000
           shape1      shape2
shape1 0.02115524 0.011131107
shape2 0.01113111 0.008749516
> summary(f1)
Fitting of the distribution ' oibeta ' by maximum likelihood
Parameters :
       estimate Std. Error
shape1 3.032457 0.14544840
shape2 2.029166 0.09353885
p1     0.167000 0.37297587
Loglikelihood:  -252.4742   AIC:  510.9485   BIC:  525.6718
Correlation matrix:
          shape1    shape2 p1
shape1 1.0000000 0.8181583  0
shape2 0.8181583 1.0000000  0
p1     0.0000000 0.0000000  1
```

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ASSESSING BIAS AND VARIANCE – ONE-INFLATED DISTR.

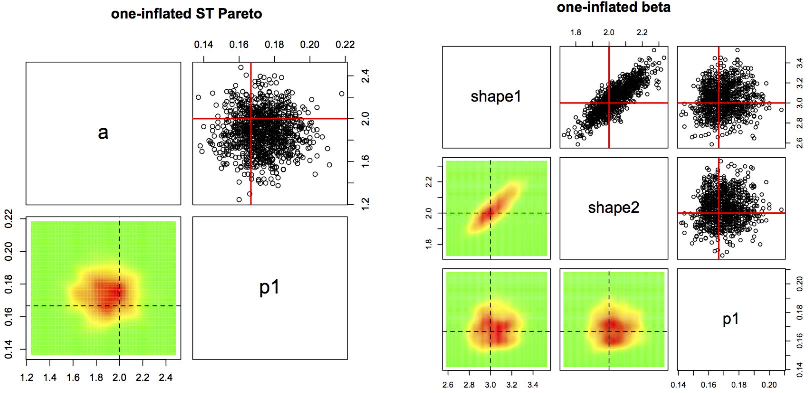

FIGURE: Bootstrap estimate of ML Estimators for oiPareto and oibeta, sample size $n = 1000$, bootstrap size $b = 1000$: function `bootdist` on `fitDR` outputs

ASSESSING BIAS AND VARIANCE – MBBEFD(A,B)

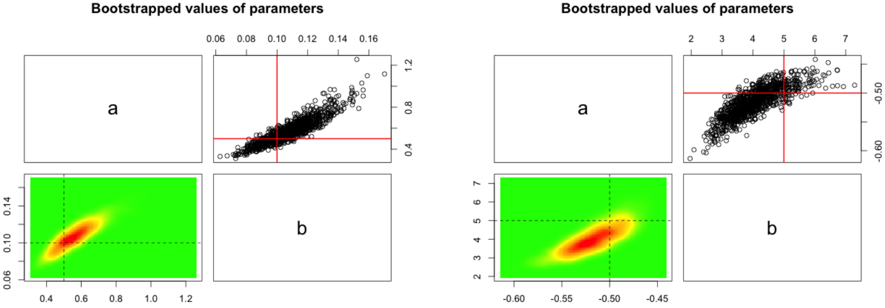

FIGURE: Bootstrap estimate of ML Estimators for MBBEFD(a,b), sample size $n = 1000$, bootstrap size $b = 1000$ for domain $\mathcal{D}^1_{a,b}$ (left) and $\mathcal{D}^2_{a,b}$ (right)

LOSSALAE DATASET – DISTRIBUTION FUNCTION

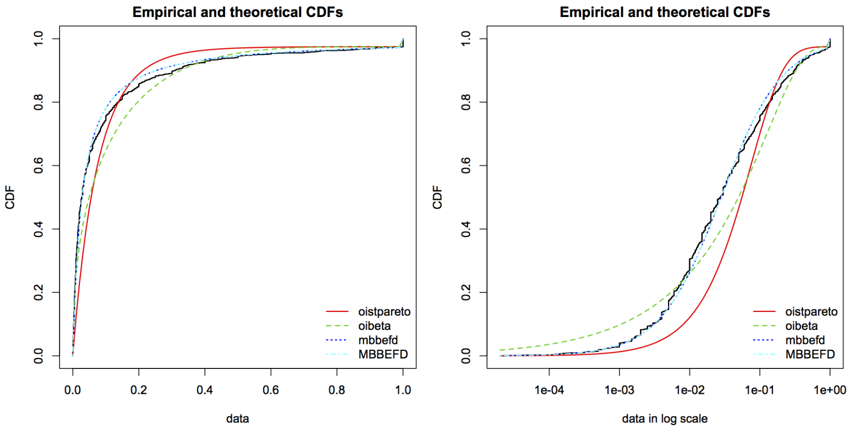

FIGURE: Fitted cdf on `lossalae`: function `cdfcomp` on `fitDR` outputs

LOSSALAE DATASET – PPPLOT AND ECPLOT

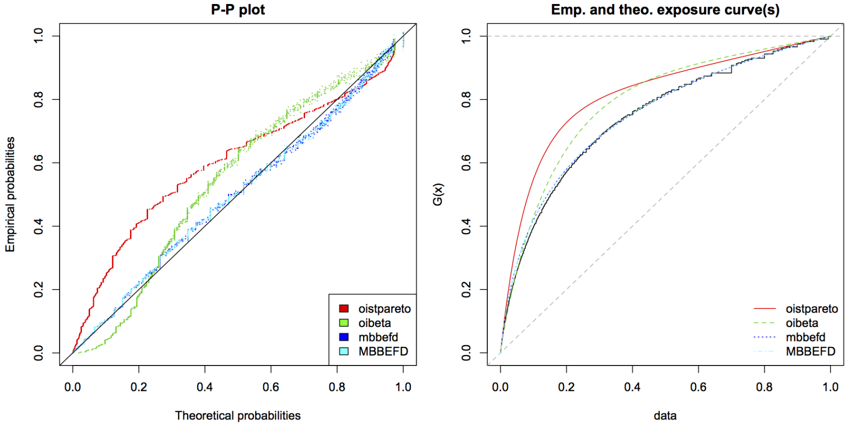

FIGURE: PP-plot and fitted densities on `lossalae`: functions `ppcomp`, `eccomp` on `fitDR` outputs

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Conclusion and Perspectives

- Fitting one-inflated distribution is carried out in a two-step procedure.
  1. estimate p_1 and compute the set of observations < 1,
  2. estimate other parameters on the other set.

- Fitting MBBEFD distribution is rather hard: use a three-step procedure.
  1. compute prefitting values based on parameter transformation,
  2. estimate parameters on subsets $\mathcal{D}^1$, $\mathcal{D}^2$
  3. take the most likely parameters.

- R package development:
  1. **mbbefd** [SDG16] for `d`, `p`, `q`, `r` functions of new distributions; computing exposure curve (theo. and emp.), fitting function `fitDR` inheriting from the `fitdist` class.
  2. use **fitdistrplus** [DMD16] for the fitting process.

- Regression models : why not consider explanatory variables?
  - dataset `asiacomrisk` contains large commercial losses caused by man-made risks with variables (usage, country,...).
  - how to use explanatory variables? in estimating the total loss probability?

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